

Infinite-Horizon Competitive Programs Are Optimal

Swapan Dasgupta and Tapan Mitra

Received April 9, 1998; revised version received September 28, 1998

For infinite-horizon optimal-growth problems the standard result in the literature says that a program is optimal if and only if associated with it is a sequence of present-value prices at which the program satisfies (i) a set of myopic “competitive conditions,” and (ii) an asymptotic “transversality condition.” The principal result of this paper points out the interesting and surprising fact that at least for a class of multisector models where the production side is described by a simple linear model, and there are some limiting primary factors, the competitive conditions alone characterize an optimal program.

Keywords: optimal programs, competitive conditions, transversality condition, inter-temporal decentralization.

JEL classification: C61, D90, O41.

1 Introduction

Ever since Malinvaud’s (1953) classic paper it is recognized that resource-allocation problems over time, with no terminal date, are different from finite-horizon or static allocation problems in an essential respect. In the latter, an allocation of resources is optimal if and only if there is an associated set of shadow prices (dual variables) at which the quantities in the allocation are also solutions to individual optimization problems (profit or utility maximization under suitable constraints) treating these prices as (parametric) competitive market prices. In this sense the plan can be decentralized or regarded as being “competitive.” This is, however, only partly the case in the former.

In this paper we will address infinite-horizon optimal-growth problems of the Ramsey type in multisector production models (see McKenzie, 1986, for a comprehensive survey). In the context of such problems, the standard

result in the literature says that a program is optimal (maximizing a discounted or undiscounted sum of utilities) if and only if associated with it is a sequence of present-value prices at which the program satisfies two types of conditions: (1) a set of "competitive conditions" which are essentially intertemporal profit and utility maximization (see Gale and Sutherland, 1968) period by period; and (2) some appropriate kind of "transversality condition," namely, the present value of capital stock asymptotically converges to zero (discounted case) or is bounded (undiscounted case). The former condition is thought of as capturing the notion of finite-horizon optimality; that is, optimality amongst the paths with a terminal date and prescribed initial and terminal stocks (see Dorfman et al., 1958). It is the latter which is supposed to capture the infinite-horizon nature of the problem, signaling capital overaccumulation along programs which are competitive or finite-horizon optimal but fail to satisfy the asymptotic condition. While the former corresponds to individual (or myopic) optimization in competitive markets and is analogous to the characterization in static problems, the latter does not seem to correspond to any myopic-behavior rule for an individual decision maker and in that sense is not decentralizable (see Koopmans, 1957; Majumdar, 1988).

The principal result of this paper points out the interesting and surprising fact that at least in a class of such multisector optimal-growth models where the production side is described by a simple linear model, where there is no joint production, there is one process for each good, and there are some limiting primary factors, the competitive conditions alone characterize an optimal program.¹ They capture both the finite- and infinite-horizon aspects of the problem. Finite-horizon optimality for every finite horizon also guarantees optimality in the infinite-horizon problem. This is because the competitive conditions for an infinite program can be shown to imply that the transversality condition is also automatically satisfied.

This is especially of interest in the context of decentralization and planning, and recent literature providing characterizations of infinite-horizon optimal paths in which the transversality condition (an asymptotic condition) is replaced by an equivalent finite-horizon condition which may be verified period by period by myopic agents. More precisely, in models

1 Actually, this result holds for a wider class of multisector models, as is clear from the more extensive analysis carried out in Dasgupta and Mitra (1991). In particular, for von Neumann models which allow for joint production, the result can be established under a stronger form of the productivity assumption than Assumption 3 used in this paper.

where there is a stationary optimal program it can be shown that a program is optimal if and only if it satisfies the competitive condition (1), together with an additional myopic condition which roughly says that (3), the deviation of prices and quantities along the competitive program, from those along the stationary optimal program, are of opposite signs in each period (see Brock and Majumdar, 1988; Dasgupta and Mitra, 1988). In the class of problems dealt with in the present paper, only the myopic competitive conditions turn out to be essential for optimality. No other auxiliary conditions, such as (3), which are difficult to interpret from the standpoint of decentralization and individual incentive compatibility, are needed. As a consequence, any additional information required in some of these characterizations, like that of a stationary optimal solution and its supporting prices, become unnecessary as well.

In brief then, this paper deals with price characterization of optimal paths in an infinite-horizon multisector optimal-growth model of the Ramsey type where the production side of the model is described by a Leontief model. As such, this is a special case of general versions of such a dynamic-optimization problem dealt with in the literature (see, e.g., Peleg, 1970, 1974; Weitzman, 1973; McKenzie, 1986) and the general characterization results, well-known in the literature, of course apply; in particular, with convex structure, it is a necessary condition for optimality of a given program that it satisfies the competitive conditions. In this paper, therefore, we focus only on conditions which are sufficient to ensure that a program is optimal. We may succinctly describe the two kinds of existing sufficiency results as follows: in one kind, a typical statement would be that a competitive program is optimal if it satisfies an asymptotic transversality condition, such as (2); in the other kind, a typical statement would read: a competitive program is optimal if it satisfies a period-by-period condition, such as (3). The novelty of the present paper lies in showing that within the context of a multisector production model, such as the one dealt with here, a much sharper statement is possible, which entirely dispenses with any "transversality condition" or an auxiliary condition, such as (3). A very simple statement can be established, namely, a competitive program is optimal.

Following preliminary definitions and results in Sect. 2, the statements of our main theorems, along with supporting auxiliary propositions, are collected in Sect. 3. We supplement formal statements with informal discussions to provide the intuitive reasoning behind the results. For the interested readers, complete formal proofs are provided in the appendix.

2 Preliminaries

2.1 The Model

The model is described by a triplet (Ω, u, δ) , where Ω , a subset of $\mathbf{R}_+^n \times \mathbf{R}_+^n$, is the technology set, $u: \mathbf{R}_+^n \rightarrow \mathbf{R}$ is the period welfare function, and δ is the discount factor. Points in Ω are written as an ordered pair (x, y) , where x denotes the initial stock of inputs and y denotes the final output which can be produced with inputs x . We shall make the following assumptions on (Ω, u, δ) .

Assumption 1: There is an $n \times n$ real matrix $A = [a_{ij}]$, $i = 1, \dots, n$, $j = 1, \dots, n$ and a vector $b = (b_1, \dots, b_n)$ in \mathbf{R}^n such that for any (x, y) in $\mathbf{R}^n \times \mathbf{R}^n$, $(x, y) \in \Omega$ iff $x \geq Ay$, $x \geq 0$, $y \geq 0$, and $by \leq 1$.²

Assumption 2: $a_{ij} \geq 0$ for all $i, j = 1, \dots, n$; for each $j = 1, \dots, n$, there is some $i = 1, \dots, n$ such that $a_{ij} > 0$; $b \gg 0$.

Assumption 3: A is productive; that is, there is $\hat{y} \gg 0$ such that $\hat{y} \gg A\hat{y}$ and $b\hat{y} \leq 1$.

Assumption 4: $u: \mathbf{R}_+^n \rightarrow \mathbf{R}$ is continuous.

Assumption 5: $0 < \delta \leq 1$.

Here, a_{ij} and b_j are, respectively, the amounts of the i -th good and labor which are required to produce one unit of output of the j -th good. Assumption 2 simply says that each production process requires a positive amount of the primary factor, as well as a positive amount of some produced factor. Assumption 3 essentially excludes the economically uninteresting case of a production system which is unable to sustain some positive consumption levels for all of the desired goods. A different interpretation of this condition is that it ensures that the series of the "direct and indirect" input requirements, $\sum_0^\infty A^t$, adds up to a finite sum. This is well-known (see Dorfman et al., 1958; Gale, 1960) and we state it formally as a remark, for future reference.

² For x, y in \mathbf{R}^n , $x \geq y$ means $x_i \geq y_i$ for $i = 1, \dots, n$; $x > y$ means $x \geq y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for $i = 1, \dots, n$. For x in \mathbf{R}^n , the sum norm of x (denoted by $\|x\|$) is defined by $\|x\| = \sum_{i=1}^n |x_i|$.

Remark 1: If A is productive and $A \geq 0$ then $A^t \rightarrow 0$ as $t \rightarrow \infty$, and $(I - A)^{-1} = \sum_0^\infty A^t$, where, by convention, A^0 is the identity matrix I .

It is clear that there is a number λ satisfying $0 < \lambda < 1$ and $A\hat{y} \ll \lambda\hat{y}$, where \hat{y} is the vector whose existence is asserted in Assumption 3. From now on we fix the number λ and the vector \hat{y} . In what follows, we normalize $u(0) = 0$.

2.2 Programs

A program from \tilde{y} in \mathbf{R}_+^n is a sequence $\langle x(t), y(t), c(t) \rangle$ such that $y(0) = \tilde{y}$, and $(x(t), y(t+1)) \in \Omega$, $c(t) \equiv y(t) - x(t) \geq 0$ for all $t \geq 0$.³

Because of the presence of a limitational primary factor, labor programs from \tilde{y} are uniformly norm-bounded by a number which depends only on the initial stock \tilde{y} .⁴ More precisely, there is a number $\beta = \max\{\|\tilde{y}\|, 1/[\min_i\{b_i\}]\}$, such that if $\langle x(t), y(t), c(t) \rangle$ is a program from \tilde{y} in \mathbf{R}_+^n then $\|y(t)\| \leq \beta$ for all $t \geq 0$ and, therefore, $\|x(t)\| \leq \beta$, $\|c(t)\| \leq \beta$, as well, for $t \geq 0$; consequently, the utility in each period, $u(c(t))$, can at most be $\max_{\|c\| \leq \beta} u(c)$, which we shall denote by B .

We now note a useful property of programs, which exploits the specific structure of the simple linear model of production. Consider the consumption in period t , $c(t)$. To produce this we need inputs in the previous period equal to $Ac(t)$. However, these inputs themselves have to be produced one period earlier, at $(t - 1)$, and require inputs equal to $A[Ac(t)] = A^2c(t)$ in period $(t - 2)$. The pattern is clear; if we go back to some earlier period T then we must have stocks at hand equal to $A^{t-T}c(t)$, which are solely devoted to generating the intermediate inputs in the intervening periods from $T + 1$ to $t - 1$, in order to be able to produce the final output $c(t)$ in period t . This is true of the consumption in each and every period t subsequent to T . The stocks at hand in period T , $y(T)$, must be large enough to provide the consumption in period T and the sum total of the inputs required to produce the future stream of consumption; that is, $y(T) \geq \sum_{t=T}^\infty A^{T-t}c(t)$. In the argument above, we allowed for the possibility that there may be

3 The expression "a program from \tilde{y} " actually means "a program starting from the initial stock \tilde{y} ." This being understood and standard in the literature, the shorter expression is used for convenience.

4 This boundedness result is true for more general models; see, for example, Dasgupta and Mitra (1988). However, in the present paper, one can provide explicit bounds for output in terms of the parameters defining the technology set.

waste in the productive process: the inputs used may be larger than the minimum necessary to produce the output. If, however, production within each period takes place in an efficient manner then the stocks at hand at time T are just enough to generate the future stream of consumption, so that $y(T) = \sum_{t=T}^{\infty} A^{t-T} c(t)$. This convenient property is quite crucial to our analysis, and we state it formally below.

Lemma 1: If $\langle x(t), y(t), c(t) \rangle$ is a program from \tilde{y} in \mathbf{R}_+^n , then for each $T \geq 0$,

$$\sum_{t=T}^{\infty} A^{t-T} c(t) \leq y(T). \quad (1a)$$

Furthermore, if $Ay(t+1) = x(t)$ for $t \geq 0$, then

$$\sum_{t=T}^{\infty} A^{t-T} c(t) = y(T). \quad (1b)$$

2.3 Optimal and Competitive Programs

An optimal program from \tilde{y} in \mathbf{R}_+^n is a program $\langle x^*(t), y^*(t), c^*(t) \rangle$ from \tilde{y} such that

$$\limsup_{T \rightarrow \infty} \sum_{t=1}^T \delta^t [u(c(t)) - u(c^*(t))] \leq 0$$

for every program $\langle x(t), y(t), c(t) \rangle$ from \tilde{y} .

In the discounted case ($0 < \delta < 1$), it is clear that for every program $\langle x(t), y(t), c(t) \rangle$ from \tilde{y} in \mathbf{R}_+^n , $\sum_0^{\infty} \delta^t u(c(t))$ is absolutely convergent. Then, the criterion of optimality of $\langle x^*(t), y^*(t), c^*(t) \rangle$ given above is equivalent to

$$\sum_0^{\infty} \delta^t u(c(t)) \leq \sum_0^{\infty} \delta^t u(c^*(t))$$

for every program $\langle x(t), y(t), c(t) \rangle$ from \tilde{y} .

In the undiscounted case ($\delta = 1$), the utility sums are not necessarily convergent, and the criterion of optimality is in terms of the familiar "overtaking criterion" (see Gale, 1967).

A competitive program from \tilde{y} in \mathbf{R}_+^n is a sequence $\langle x(t), y(t), c(t) \rangle$,

$p(t)$ such that $\langle x(t), y(t), c(t) \rangle$ is a program from \tilde{y} , $p(t)$ is in \mathbf{R}_+^n for $t \geq 0$, and the inequalities below hold:

$$\delta^t u(c(t)) - p(t)c(t) \geq \delta^t u(c) - p(t)c \quad \text{for all } c \geq 0, t \geq 0, \quad (2)$$

$$p(t+1)y(t+1) - p(t)x(t) \geq p(t+1)y - p(t)x$$

for all $(x, y) \in \Omega, t \geq 0$. (3)

Given a competitive program $\langle x(t), y(t), c(t), p(t) \rangle$ from \tilde{y} , we interpret $p(t)$ as the present-value price vector at date t , reflecting the shadow valuation of the n producible goods in the economy at that date.

We associate with a competitive program $\langle x(t), y(t), c(t), p(t) \rangle$ from \tilde{y} , a sequence $\langle w(t) \rangle$, where $w(t)$ is in \mathbf{R}_+ for $t \geq 0$, and $w(t)$ is defined by:

$$w(t) = p(t+1)y(t+1) - p(t)x(t) \quad \text{for } t \geq 0. \quad (4)$$

We interpret $w(t)$ as the present-value wage at date t , reflecting the shadow valuation of labor (primary factor) at that date.

The interpretations of the inequalities defining a competitive program are standard. The first inequality implies utility maximization over consumption bundles which are possible to purchase at the prices $p(t)$ and with the same expenditure as the bundle consumed, $c(t)$.

The second inequality implies that present-value profit (present-value revenue minus input cost) is maximized over the set of feasible production processes in each period. Since we are dealing with a linear-activities model, at the prices $p(t), w(t)$, each activity at best breaks even (value of output from the activity does not exceed the cost of production, including, within the cost of production, the cost of produced inputs as well as that of the primary factor, labor). Activities which are used at a positive level do break even (show zero profit) while activities which are not in use may show a loss. Further, factors of production, both produced as well as primary factors, which are in surplus, have zero value. All this is formally stated in the following result: the proof is clear from the literature on duality theory of price supports for efficient allocation of resources in static models, and is therefore omitted.

Lemma 2: If $\langle x(t), y(t), p(t), c(t) \rangle$ is a competitive program from \tilde{y} in \mathbf{R}_+^n , with an associated sequence $\langle w(t) \rangle$ satisfying (4), then

- i. $0 = [p(t+1) - p(t)A - w(t)b]y(t+1) \geq [p(t+1) - p(t)A - w(t)b]y$ for all $t \geq 0$, and for all y in \mathbf{R}_+^n ;

- ii. if $y(t+1) \gg 0$ for some $t \geq 0$, then $[p(t+1) - p(t)A - w(t)b] = 0$ for that t ;
- iii. if $p(t) \gg 0$ for some $t \geq 0$, then $x(t) = Ay(t+1)$ for that t ;
- iv. if $w(t) > 0$ for some $t \geq 0$, then $by(t+1) = 1$ for that t .

3 Optimality of Competitive Programs

3.1 Competitive Valuation of Labor

In this section we establish a property regarding the long-run valuation of labor (primary factor) along a competitive program. This property constitutes the principal technical result of the paper, enabling us to establish the optimality of competitive programs (in the next two sections). However, it is clearly also of independent interest.

Given a competitive program $\langle x(t), y(t), c(t), p(t) \rangle$ from \tilde{y} in \mathbf{R}_+^n , with associated (present value of wage) sequence $\langle w(t) \rangle$, we define its current-value price sequence $\langle q(t) \rangle$ and its current-value wage sequence $\langle v(t) \rangle$ by $q(t) = [p(t)/\delta^t]$, $v(t) = [w(t)/\delta^t]$ for $t \geq 0$.

A basic property of competitive programs is that the current-value wage sequence is uniformly bounded above. To establish this property we will use the following additional assumption on the utility function.

Assumption 6: (i) If $u(c) > 0$ and $c' > c$, then $u(c') > u(c)$; (ii) $u(v\varepsilon)/v \rightarrow \infty$ as $v \rightarrow 0$; (iii) if $u(c) > 0$ and $c_i = 0$ for some i in $\{1, \dots, n\}$, then $[u(c + \varepsilon e^i) - u(c)]/\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$ for that i , where e^i is the i -th unit vector in \mathbf{R}^n .

Assumption 6 states restrictions on the utility function which are fairly standard. Part i simply says that the produced goods are desirable: utility is increasing in each good. This property is assumed to hold when utility is positive (recall that the utility of the zero-consumption bundle has been normalized to zero) so as to permit the inclusion of the standard example of a Cobb–Douglas function (where utility is zero along the boundary of the nonnegative orthant).

Parts ii and iii of Assumption 6 have the same function as the well-known Inada conditions. They essentially say that the rate of increase of utility is large when consumption levels are small. They, therefore, ensure that utility as well as consumption levels of all goods are positive along competitive programs and allow us to concentrate only on interior programs, where

all relevant variables are positive, ignoring programs which may reach the boundary. Dealing with the latter possibility increases the technical complexities without shedding much light on the general nature of the arguments.

Under Assumption 6, along a competitive program, consumption, output, and prices are positive at each date. Since all activities are operated at positive levels, they all break even. Since all goods prices are positive (they are scarce), there is no wastage of produced inputs. These observations are stated formally, for future reference, in the following lemma.

Lemma 3: If $\langle x(t), y(t), c(t), p(t) \rangle$ is a competitive program from \tilde{y} in \mathbf{R}_+^n , with an associated sequence $\langle w(t) \rangle$ satisfying (4), then

- i. $\langle c(t), y(t), p(t) \rangle \gg 0$ for $t \geq 0$;
- ii. $p(t + 1) = p(t)A + w(t)b$ for $t \geq 0$;
- iii. $x(t) = Ay(t + 1)$ for $t \geq 0$.

We may now state the main result of this section.

Proposition 1: Let $\langle x(t), y(t), c(t), p(t) \rangle$ be a competitive program from \tilde{y} in \mathbf{R}_+^n . Then,

$$\limsup_{t \rightarrow \infty} v(t) < \infty . \tag{5}$$

We provide a heuristic explanation of this technical result. In what follows we deal only with current prices which, as the competitive conditions show, are the marginal utilities of the goods. Write r as the interest rate, which is implicit in the discount factor δ ; that is, $\delta = 1/(1+r)$. The current price of the j -th good, in period $t + 1$, is the outlay on the inputs (both intermediate inputs and labor) in period t plus the interest on this outlay. That is, $q_j(t + 1) = (1+r)[\sum_{i=1}^n a_{ij}q_i(t) + v(t)b_j]$; or, in vector notation, $q(t + 1) = (1/\delta)[q(t)A + v(t)b]$.

If, at some period T , the current wage $v(T)$ is “very large,” then the prices of all goods in the next period, $T + 1$, will be large, because of the large labor cost of production. As a result, the cost of (produced) inputs of the output in period $T + 2$, i.e., $q(T + 1)A$ will be large and, consequently, prices in the next period, $q(T + 2)$ will also be large. Clearly, this line of reasoning can be carried forward for a long period of time to conclude that a high level of current wage at time T will give rise to high levels of prices

of goods persisting for a long period of time N ; the higher the initial wage $v(T)$, the longer the period N and the higher the prices prevailing during this period. Along a competitive program, the current price of a good equals its marginal utility; thus, over the period N , marginal utilities must be very large, that is to say, consumption levels chosen must be very small.

Observe that, since the series of direct and indirect input requirements, $\sum_{t=T+1}^{\infty} A^{t-(T+1)}c(t)$, has a finite sum, the contribution of the consumption stream, beyond a large enough horizon N , to the initial input-stock requirement is very small. In other words, for N large, $\sum_{t=T+1}^{T+N} A^{t-(T+1)}c(t)$ is approximately equal to the initial stocks $y(T+1)$. If the consumption levels (from $T+1$ to $T+N$) are very small, the current activity levels $y(T+1)$, and the labor used in period T , would be small. Consequently, there would be surplus labor in period T and this is clearly inconsistent with the premise that labor is scarce and its current wage is very high.

3.2 Optimality of Competitive Programs: the Discounted Case

In this section, we consider only the discounted case ($0 < \delta < 1$). A competitive program $\langle x(t), y(t), c(t), p(t) \rangle$ is said to satisfy the transversality condition if $\lim_{t \rightarrow \infty} p(t)y(t) = 0$. It is well-known⁵ that a competitive program which satisfies the transversality condition is optimal. We state this formally below, for ready reference.⁶

Theorem 1: If $\langle x(t), y(t), c(t), p(t) \rangle$ is a competitive program from any \tilde{y} in \mathbf{R}_+^n and

$$\lim_{t \rightarrow \infty} p(t)y(t) = 0, \quad (6)$$

then $\langle x(t), y(t), c(t) \rangle$ is an optimal program from \tilde{y} .

In the discounted case, we can show, using Proposition 1, that for every competitive program, the present value of output is asymptotically zero,

5 While one can refer to treatments of this result in the more general "reduced-form" model by Weitzman (1973) and McKenzie (1986), it is most convenient to directly appeal to the analysis given in Dasgupta and Mitra (1990).

6 It is interesting to note that the condition $\lim_{t \rightarrow \infty} p(t)y(t) = 0$ of Theorem 1 can be replaced by the apparently weaker (but actually equivalent in this context) conditions given by $\liminf_{t \rightarrow \infty} p(t)y(t) = 0$ or $\limsup_{t \rightarrow \infty} p(t)y(t) < \infty$. For a discussion of these alternative "transversality conditions," see the analysis in Dasgupta and Mitra (1994).

and so the program is necessarily optimal. This is the main result of this section.

Theorem 2: If $\langle x(t), y(t), c(t), p(t) \rangle$ is a competitive program from any \tilde{y} in \mathbf{R}_+^n then, $\langle x(t), y(t), c(t) \rangle$ is an optimal program from \tilde{y} .

We provide an intuitive argument below in which, whenever we refer to prices, cost, wage, these are understood to be present values; that is, they refer to valuations according to the price system $p(t), w(t)$.

As we observed earlier, according to the properties of equilibrium prices in a linear model, encapsulated in the profit conditions (3), in each period, feasible production processes at best break even (see Lemma 2). By Assumption 3, there is a feasible (composite) process which allows some expansion possibility so that there is an output vector \hat{y} which is possible to produce from inputs which amount to at most a fraction λ of \hat{y} .

Let us consider the behavior of the present value of this stock $(p(t)\hat{y})$ over time. Since the process at best breaks even, the value of output \hat{y} is at most the value of the produced inputs, valued at the previous-period prices, plus the wage. However, the produced inputs are a fraction λ of \hat{y} , so the value of \hat{y} at time t is at most the wage plus a fraction λ of its value in the previous period. Similarly, the value of \hat{y} in $(t - 1)$ is at most the wage in $(t - 2)$ plus the cost of the produced inputs which are only a fraction λ of \hat{y} . Consequently, the value of \hat{y} at t decomposes into the direct wage cost $w(t - 1)$, a fraction λ of the wage cost a period earlier, $w(t - 2)$, and a fraction λ^2 of the value of \hat{y} at $(t - 2)$. The pattern is clear: the value of \hat{y} at time t may be decomposed into a sum of current and past wages and the value of \hat{y} at the initial period, each of these terms being weighted by weights which geometrically decline at the rate of λ , the further back in time being the value considered.

It is now clear that if the time t is sufficiently far in the distant future, the effect of the value of \hat{y} in the initial period, within the decomposition, becomes negligible. Similarly, the effect of the wage costs in the initial periods may also be neglected. We are, therefore, left with the effect of the wage terms in the final periods, close to the period t .

Now recall that the current wage sequence is bounded (Proposition 1); therefore, the present value of wages (the current wages discounted at the rate of r) becomes negligible in the distant future. Consequently, for large values of t , the effect of the wage terms in the final periods, close to the period t , are also negligible, thereby leading us to the conclusion that the

present-value prices of the goods, p_t , become negligibly small as t becomes large. In other words, along a competitive path, the transversality condition is satisfied and, therefore, the path is optimal (by Theorem 1).

3.3 Optimality of Competitive Programs: the Undiscounted Case

In this section we deal only with the undiscounted case ($\delta = 1$). In order to establish the optimality of competitive programs, we will use, in addition to Assumptions 1–6, the following assumption.

Assumption 7: u is strictly concave on \mathbf{R}_+^n .

The assumption that marginal utilities are decreasing (u is concave) is standard in the optimum-growth literature. We are making the assumption that u is concave, in particular, that it is strictly concave, to avoid technical complications. It is possible to get by with a weaker assumption which ensures a strict “value loss property” (including both the utility as well as production price support) at the golden-rule equilibrium (see Brock, 1970). We prefer to use the simplifying Assumption 7, as it is easy to check in applications.

The standard result on price characterization of optimal programs in the undiscounted case says that if a program is competitive and the supporting price path is bounded then the program is optimal. We first state this formally for ready reference. (Because none of the extant treatments in the literature directly apply to the model we are considering, even though their methods do apply with suitable modifications, we also provide a self-contained proof of this result in the appendix, for the convenience of the interested reader.)

Theorem 3: If $\langle x(t), y(t), c(t), p(t) \rangle$ is a competitive program from any \tilde{y} in \mathbf{R}_+^n and

$$\limsup_{t \rightarrow \infty} \|p(t)\| < \infty, \quad (7)$$

then $\langle x(t), y(t), c(t) \rangle$ is an optimal program from \tilde{y} .⁷

⁷ The condition (7) can be replaced by the condition that $\limsup_{t \rightarrow \infty} p(t)y(t) < \infty$. To see this, note that the latter condition can be used to show (using the same proof as in Theorem 3) that $c(t) \rightarrow c^*$ as $t \rightarrow \infty$, and so $y(t) \rightarrow y^*$ as $t \rightarrow \infty$

We now state the main result of this section.

Theorem 4: If $\langle x(t), y(t), c(t), p(t) \rangle$ is a competitive program from any \tilde{y} in \mathbf{R}_+^n , then $\langle x(t), y(t), c(t) \rangle$ is an optimal program from \tilde{y} .

The heuristic reasoning for this result is similar to that for Theorem 2.⁸ The difference is that, since $\delta = 1$, present and current values are the same so we need not distinguish between the two concepts.

We may now follow the argument provided for Theorem 2 to conclude that the value of the stocks \hat{y} at time t may be decomposed into a sum of current and past wages and the value of \hat{y} at the initial period, each of these terms being weighted by weights which geometrically decline at the rate of the fraction λ , the further back in time being the value considered. As before, if the time t is large then the effect of the value of \hat{y} in the initial period, within the decomposition, becomes negligible and we are left with only the sum of the weighted wage terms. By Proposition 1, the wage terms are bounded. Since the weights are geometrically declining, the contribution of the wages in the decomposition is bounded, leading us to the conclusion that the price sequence is bounded. Hence, by Theorem 3, the competitive program must be optimal.

4 Conclusion

Dynamic optimization models are now widely used in studying problems of macroeconomics, international trade, and public economics. In these models, competitive paths satisfy the well-known Ramsey–Euler equations.

(by Lemma 1). Since the golden-rule equilibrium is a competitive program, $c^* \gg 0$ and so $y^* \gg 0$ (by Lemma 3). Thus, (7) is then also satisfied. Conversely, given (7), the boundedness of output ensures that $\limsup_{t \rightarrow \infty} p(t)y(t) < \infty$. Thus, the two conditions are equivalent in this context and we can use them interchangeably.

⁸ Note that our principal technical result (Proposition 1) uses Assumption 6, which is a significant, though plausible, restriction on the welfare function. Our results on the optimality of competitive programs (Theorems 2 and 4) use Proposition 1, and so Assumption 6. However, it appears to us that it might be possible to establish Theorem 2 (the optimality of competitive programs in the discounted case) without using Assumption 6, by following an alternative approach (see, e.g., Dasgupta and Mitra, 1991). We are unsure, though, about making such a claim regarding Theorem 4 (the optimality of competitive programs in the undiscounted case). In any case, we feel that Proposition 1 is of sufficient independent interest to warrant the use of Assumption 6 in this paper.

However, in order to demonstrate that a solution to these equations also solves the social planner's problem (or the representative agent's problem) one has to verify that the solution also satisfies an asymptotic transversality condition. This paper shows that in a class of interesting dynamic-optimization models there is no need to check that this additional transversality condition holds.

Appendix

*Proof of Lemma 1:*⁹ Let $T \geq 0$ be given. For $t \geq T$, we have

$$c(t) = y(t) - x(t) \leq y(t) - Ay(t+1). \quad (8)$$

Thus, for $N \geq 0$, multiplying the inequality in (8), for each period t , by A^{t-T} we get

$$S(N) \equiv \sum_{t=T}^{T+N} A^{t-T} c(t) \leq y(T) - A^{N+1}y(T+N+1).$$

Clearly, $\langle S(N) \rangle$ is a monotonic nondecreasing sequence in N , bounded above by $y(T)$, so it converges. Since $A \geq 0$ and $y(t) \geq 0$ for all $t \geq 0$, this establishes (1a).

If $Ay(t+1) = x(t)$ for $t \geq 0$, then the inequality in (8) is replaced by an equality for each $t \geq 0$. Since $y(T+N+1)$ is norm-bounded by β while $A^{N+1} \rightarrow 0$ as $N \rightarrow \infty$ by Remark 1, $A^{N+1}y(T+N+1)$ converges to the zero vector, and we obtain (1b). \square

Proof of Lemma 3: Using the competitive condition (2) and Assumption 6.ii, we must have $u(c(t)) > 0$ for $t \geq 0$. So, using (2) again, and Assumption 6.iii, $c(t) \gg 0$ for $t \geq 0$. This implies that $y(t) \gg 0$ for $t \geq 0$ and also, by Assumption 6.i, that $p(t) \gg 0$. This establishes part i of the lemma. Parts ii and iii now follow from Lemma 2. \square

Proof of Proposition 1: Given λ and \hat{y} (see Assumption 3 and related comments), define a number θ by $\theta \equiv [\hat{B} / \min_i \{\hat{y}_i\}]$, where \hat{B} is given by $\max(B, \beta)$, and a positive integer, N , large enough so that

$$4\lambda^N [\theta / (1 - \lambda)] b \hat{y} \leq 1. \quad (9)$$

⁹ The proof of this result follows closely the line of argument used by Majumdar (1974).

Next, choose a positive number M , large enough so that

$$(4/M)[\theta/(1 - \lambda)]b\hat{y} \leq 1. \tag{10}$$

Finally, denote $(1/\delta) \min_i \{b_i\}$ by μ , $\min_{i,j} \{ (1/\delta)a_{ij} \mid a_{ij} > 0 \}$ by $\hat{\alpha}$, and $\min\{1, \hat{\alpha}\}$ by α .

If (5) is violated, we can find a positive integer T , such that

$$v(T) \geq (1/\mu)(M/\alpha^N). \tag{11}$$

By Lemma 3, $p(t + 1) = p(t)A + w(t)b$ for $t \geq 0$, which we can rewrite as

$$q(t + 1) = q(t)(1/\delta)A + v(t)(1/\delta)b \quad \text{for } t \geq 0. \tag{12}$$

Denoting $\mu v(T)$ by k , we get from (12) that

$$q_i(T + 1) \geq k \quad \text{for } i = 1, \dots, n, \tag{13}$$

so that $q(T + 1) \geq ke$, where $e = [1, \dots, 1]$ in \mathbf{R}^n . Using (12) repeatedly, we will then obtain

$$q(t + 1) \geq \alpha^{t-T}ke \quad \text{for } t \geq T. \tag{14}$$

Thus, using (11) we get

$$q(T + s) \geq Me \quad \text{for } s = 1, \dots, N. \tag{15}$$

Using the competitive condition (2), we have $q(t)c(t) \leq u(c(t))$ for $t \geq 0$, so that (15) yields $M\|c(t)\| \leq q(t)c(t) \leq u(c(t)) \leq B$ for $t = T + 1, \dots, T + N$, and

$$\|c(t)\| \leq [B/M] \quad \text{for } t = T + 1, \dots, T + N. \tag{16}$$

By Lemma 3, we have $x(t) = Ay(t + 1)$ for $t \geq 0$. Using Lemma 1, we obtain

$$\sum_{t=T+1}^{\infty} A^{t-(T+1)}c(t) = y(T + 1). \tag{17}$$

We proceed now to put bounds on the left-hand-side expression of (17) by using the bounds on consumption given by (16). We know that $c(t) \leq \beta e \leq \theta\hat{y}$ for all $t \geq 0$, so, in particular, $A^{t-(T+1)}c(t) \leq \theta\lambda^{t-(T+1)}\hat{y}$ for

$t \geq T + N + 1$, and

$$\sum_{t=T+N+1}^{\infty} A^{t-(T+1)} c(t) \leq \lambda^N [\theta/(1-\lambda)] \hat{y}. \quad (18)$$

Also, by using (16) we have for $t = T + 1, \dots, T + N$, $c(t) \leq (B/M)e \leq (\theta/M) \hat{y}$, so that $A^{t-(T+1)} c(t) \leq (\theta/M) \lambda^{t-(T+1)} \hat{y}$, and

$$\sum_{t=T+1}^{T+N} A^{t-(T+1)} c(t) \leq (1/M) [\theta/(1-\lambda)] \hat{y}. \quad (19)$$

Combining (17), (18), and (19), we get

$$y(T+1) \leq \lambda^N [\theta/(1-\lambda)] \hat{y} + (1/M) [\theta/(1-\lambda)] \hat{y}. \quad (20)$$

Multiplying through by b ,

$$by(T+1) \leq \lambda^N [\theta/(1-\lambda)] b \hat{y} + (1/M) [\theta/(1-\lambda)] b \hat{y}. \quad (21)$$

Using (9) and (10) in (21), $by(T+1) \leq (1/2)$ so that, by Lemma 2, $w(T) = 0$ and $v(T) = 0$, contradicting (11). \square

Proof of Theorem 2: By Lemma 3, we have

$$p(t+1) = p(t)A + w(t)b \quad \text{for } t \geq 0. \quad (22)$$

By Assumption 3, we are given λ and \hat{y} such that $0 < \lambda < 1$ and $\lambda \hat{y} \gg A \hat{y}$. Multiplying (22) by \hat{y} , and noting that $b \hat{y} \leq 1$, we get

$$p(t+1) \hat{y} \leq \lambda p(t) \hat{y} + w(t). \quad (23)$$

Iterating on the inequality (23),

$$p(t+1) \hat{y} \leq \lambda^{t+1} p(0) \hat{y} + \sum_{s=0}^t \lambda^{t-s} w(s). \quad (24)$$

Using Proposition 1, we can find $0 < M < \infty$, such that $v(t) = [w(t)/\delta^t] \leq M$ for $t \geq 0$. Thus, (24) yields

$$p(t+1) \hat{y} \leq \lambda^{t+1} p(0) \hat{y} + \sum_{s=0}^t \lambda^{t-s} \delta^s M. \quad (25)$$

Denoting $\max\{\lambda, \delta\}$ by ρ , we note that $0 < \rho < 1$, and (25) yields

$$p(t + 1)\hat{y} \leq \rho^{t+1}p(0)\hat{y} + M(t + 1)\rho^t. \tag{26}$$

Since the right-hand-side of (26) converges to zero as $t \rightarrow \infty$, we obtain (6). Using Theorem 1, $(x(t), y(t), c(t))$ is an optimal program from \tilde{y} . \square

Our proof of Theorem 3 will require several steps. First, we will establish a familiar “golden-rule equilibrium” (Proposition 2). Second, we will establish the asymptotic behavior of programs which are “good” (Proposition 3). Third, we will check that “good” and “bad” programs do indeed exhaust the class of feasible programs (Proposition 4). Finally, we will establish that a competitive program with uniformly bounded prices is optimal.¹⁰

A golden-rule equilibrium is a vector (x^*, y^*, c^*, p^*) in \mathbf{R}_+^{4n} such that

- i. $(x^*, y^*) \in \Omega$ with $x^* = Ay^*$; $c^* = y^* - x^*$;
- ii. $u(c^*) - p^*c^* \geq u(c) - p^*c$ for all $c \in \mathbf{R}_+^n$;
- iii. $p^*y^* - p^*x^* \geq p^*y - p^*x$ for all $(x, y) \in \Omega$.

Proposition 2: There exists a golden-rule equilibrium.

Proof: Consider the following constrained-maximization problem:

$$\{\max u(z - Ay) \text{ subject to } y - z \geq 0, (z, y) \in \Omega\} = (\text{MAX}).$$

Denoting $(1/[\min_i \{b_i\}])$ by β_0 , we have $\|y\| \leq \beta_0$, and so the constraint set is compact. It is clearly nonempty. Using the continuity of u , (MAX) has a solution, call it (z^*, y^*) . Denoting $(z^* - Ay^*)$ by c^* , we note that if (\bar{z}, \bar{y}) is any solution to (MAX), then $\bar{z} - A\bar{y} = c^*$ by Assumption 7. Using Assumptions 3 and 6, we know that $u(c^*) \geq u(\hat{y} - A\hat{y}) > 0$. Thus, we must have, using Assumption 6 again, $y^* = z^*$ and $\bar{y} = \bar{z}$. Consequently, $(I - A)y^* = y^* - Ay^* = z^* - Ay^* = c^* = \bar{z} - A\bar{y} = \bar{y} - A\bar{y} = (I - A)\bar{y}$. Since $(I - A)^{-1}$ exists (see Remark 1), we must have $y^* = \bar{y}$ and $z^* = \bar{z}$. Thus, the solution to (MAX) is unique. Denoting Ay^* by x^* , we refer to (x^*, y^*, c^*) as a golden rule.

¹⁰ Our approach is to use the technique of Gale (1967). Specifically, the first two steps follow the analysis of Gale, modified suitably to apply to a model in which welfare is derived solely from consumption, as in Peleg (1974). In the third step, we cannot use any form of “strict convexity” of the technology set, given the linear model, unlike the treatments in Peleg (1974), Brock and Majumdar (1988), and others. The analysis here relies heavily on the specific structure of the technology set and is somewhat similar to that used in Mitra and Wan (1986) in their study of forest-management problems.

Using Assumption 3 to satisfy Slater's condition, we can apply the Kuhn–Tucker theorem to obtain p^* in \mathbf{R}_+^n such that for all (z, y) in Ω ,

$$u(z - Ay) + p^*(y - z) \leq u(c^*) . \quad (27)$$

If we pick any y in \mathbf{R}_+^n , satisfying $by \leq 1$, and define $z = Ay + c^*$, then $(z, y) \in \Omega$, and (27) yields $p^*(y - Ay - c^*) \leq 0$, so that $p^*y - p^*Ay \leq p^*y^* - p^*x^*$. Thus if x is any vector in \mathbf{R}_+^n satisfying $(x, y) \in \Omega$, then since $x \geq Ay$, we have

$$p^*y - p^*x \leq p^*y^* - p^*x^* . \quad (28)$$

Next, given any c in \mathbf{R}_+^n , if we define $y = y^*$, $z = c + Ay^*$, we have $(z, y) \in \Omega$, and applying (27), we get $u(c) + p^*(y^* - c - Ay^*) \leq u(c^*)$, so that

$$u(c) - p^*c \leq u(c^*) - p^*c^* . \quad (29)$$

Using (28) and (29), we can conclude that (x^*, y^*, c^*, p^*) is a golden-rule equilibrium. \square

Using (29) and Assumption 7, we get

$$u(c^*) - p^*c^* > u(c) - p^*c \quad \text{for } c \text{ in } \mathbf{R}_+^n, c \neq c^* . \quad (30)$$

Thus, if we define, for c in \mathbf{R}_+^n , the function $d(c, c^*) = [u(c^*) - p^*c^*] - [u(c) - p^*c]$ we note that $d(c, c^*) > 0$ when $c \neq c^*$, and $d(c, c^*) = 0$ when $c = c^*$.

Consider any $\varepsilon > 0$ for which the set

$$F(\varepsilon) \equiv \{c \text{ in } \mathbf{R}_+^n \mid \|c - c^*\| \geq \varepsilon, \|c\| \leq \beta\} \quad (31)$$

is nonempty. Then, since $F(\varepsilon)$ is compact, and $d(c, c^*)$ is continuous on $F(\varepsilon)$, it attains a minimum at some $\tilde{c}(\varepsilon)$ in $F(\varepsilon)$. Using (30), $d(\tilde{c}(\varepsilon), c^*) > 0$. Defining $d(\varepsilon) \equiv d(\tilde{c}(\varepsilon), c^*)$,

$$d(c, c^*) \geq d(\varepsilon) > 0 \quad \text{for all } c \text{ in } F(\varepsilon) . \quad (32)$$

Following Gale (1967), let us call a program $\langle x(t), y(t), c(t) \rangle$ good if there is a real number G , such that for all $T \geq 0$, we have

$$\sum_{t=0}^T [u(c(t)) - u(c^*)] \geq G . \quad (33)$$

We call a program $\langle x(t), y(t), c(t) \rangle$ bad if

$$\sum_{t=0}^T [u(c(t)) - u(c^*)] \rightarrow -\infty \text{ as } T \rightarrow \infty. \quad (34)$$

Proposition 3: If $\langle x(t), y(t), c(t) \rangle$ is a good program from \tilde{y} , then

$$\lim_{t \rightarrow \infty} c(t) = c^*. \quad (35)$$

Proof: If $\langle x(t), y(t), c(t) \rangle$ is a good program from \tilde{y} in \mathbf{R}_+^n , then using (28) and (29), and the definition of d , we can write

$$\sum_{t=0}^T [u(c(t)) - u(c^*)] \leq p^*(\tilde{y} - y^*) + p^*(x^* - x(T)) - \sum_{t=0}^T d(c(t), c^*). \quad (36)$$

Combining (33) and (36), we have for $T \geq 0$,

$$S(T) \equiv \sum_{t=0}^T d(c(t), c^*) \leq p^*\tilde{y} + p^*x^* - G. \quad (37)$$

The sequence $\langle S(T) \rangle$ is monotonically nondecreasing and bounded above, so it converges. That is, $\sum_{t=0}^{\infty} d(c(t), c^*) < \infty$, so that $d(c(t), c^*) \rightarrow 0$ as $t \rightarrow \infty$. Using (32) it follows that (35) must hold. \square

Proposition 4: If $\langle x(t), y(t), c(t) \rangle$ is a program from \tilde{y} which is not good, then it is bad.

Proof: Given any real number L , we will show that there exists a positive integer H such that

$$\sum_{t=0}^T [u(c(t)) - u(c^*)] \leq L \quad \text{for all } T \geq H. \quad (38)$$

Denote $2\beta\|p^*\|$ by G_1 . Since $\langle x(t), y(t), c(t) \rangle$ is not good, we can find H large enough so that

$$\sum_{t=0}^{H-1} [u(c(t)) - u(c^*)] \leq (L - G_1). \quad (39)$$

Using the conditions (28) and (29), we obtain for $N \geq 1$,

$$\sum_{t=H}^{H+N} [u(c(t)) - u(c^*)] \leq p^*(y(H) - y^*) + p^*(x^* - x(H + N)) \tag{40}$$

$$\leq p^*(y(H) + x^*) .$$

Since $\|y(H)\| \leq \beta$ and $\|x^*\| \leq \beta$, we have $y(H) \leq \beta e$ and $x^* \leq \beta e$, so that $p^*(y(H) + x^*) \leq 2\beta \|p^*\| = G_1$. Thus, for all $N \geq 1$, using (39) and (40), $\sum_{t=0}^{H+N} [u(c(t)) - u(c^*)] \leq L$ which establishes (38). \square

Proof of Theorem 3: Suppose, on the contrary, there is a program $\langle x'(t), y'(t), c'(t) \rangle$ from \tilde{y} , a number $\varepsilon > 0$, and a sequence $\langle T_s \rangle$, such that for $s = 1, 2, 3, \dots$

$$\sum_{t=0}^{T_s} [u(c'(t)) - u(c(t))] \geq \varepsilon . \tag{41}$$

For any $T \geq 0$, we have $\sum_{t=0}^T [u(c'(t)) - u(c(t))] = \sum_{t=0}^T [u(c'(t)) - u(c^*)] + \sum_{t=0}^T [u(c^*) - u(c(t))]$. Using the competitive conditions (2) and (3) for $(x, y) = (x^*, y^*)$ and $c = c^*$ for $t = 0, \dots, T$, $\sum_{t=0}^T [u(c^*) - u(c(t))] \leq p(0)(y^* - \tilde{y}) + p(T)(x(T) - x^*) \leq p(0)y^* + p(T)x(T)$. By (7), there is a number α such that $\|p(T)\| \leq \alpha$ for $t \geq 0$. Using this, we have $p(T)x(T) \leq \|p(T)\| \|x(T)\| \leq \alpha\beta$, so that for all $T \geq 0$,

$$\sum_{t=0}^T [u(c^*) - u(c(t))] \leq p(0)y^* + \alpha\beta , \tag{42}$$

establishing that $\langle x(t), y(t), c(t) \rangle$ is good. Using (41) and (42), we can conclude that for the sequence $\langle T_s \rangle$, we have $\sum_{t=0}^{T_s} [u(c'(t)) - u(c^*)] \geq \varepsilon - p(0)y^* - \alpha\beta$. This establishes, by using Proposition 4, that $\langle x'(t), y'(t), c'(t) \rangle$ is also good.

Using Proposition 3, we then have

$$\lim_{t \rightarrow \infty} c(t) = c^* ; \quad \lim_{t \rightarrow \infty} c'(t) = c^* . \tag{43}$$

By Assumption 3, we are given λ and \hat{y} such that $0 < \lambda < 1$ and $\lambda \hat{y} \gg A \hat{y}$. Define the number η by $\eta \equiv [\varepsilon(1 - \lambda)/4\alpha\beta]$. Using (43), we can find N sufficiently large such that for $t \geq N$,

$$c^* - \eta \hat{y} \leq c(t) \leq c^* + \eta \hat{y} ; \quad c^* - \eta \hat{y} \leq c'(t) \leq c^* + \eta \hat{y} . \tag{44}$$

Pick s sufficiently large such that $T_s > N$; denote this T_s by T to ease the notation. Then, using the competitive conditions (2) and (3),

$$\sum_{t=0}^T [u(c'(t)) - u(c(t))] \leq p(T)[x(T) - x'(T)]. \quad (45)$$

Using (41) and (45), we have $p(T)[x(T) - x'(T)] \geq \varepsilon$. Thus $p(T)[y(T) - y'(T)] = p(T)[x(T) - x'(T)] + p(T)[c(T) - c'(T)] \geq \varepsilon - 2\eta p(T)\hat{y} \geq \varepsilon - 2\eta\alpha\beta$. By choice of η , $2\eta\alpha\beta = \varepsilon(1 - \lambda)/2$, so that

$$p(T)[y(T) - y'(T)] \geq (\varepsilon/2) + (\varepsilon\lambda/2). \quad (46)$$

Since $\langle x(t), y(t), c(t), p(t) \rangle$ is competitive, by Lemma 3, $Ay(t + 1) = x(t)$ for $t \geq 0$. Thus, Lemma 1 yields $\sum_{t=T}^{\infty} A^{t-T} c(t) = y(T)$, and $\sum_{t=T}^{\infty} A^{t-T} c'(t) \leq y'(T)$. Then, $p(T)y(T) = p(T)[\sum_{t=T}^{\infty} A^{t-T} c(t)]$; and $p(T)y'(T) \geq p(T)[\sum_{t=T}^{\infty} A^{t-T} c'(t)]$. Thus, using (44), $p(T)(y(T) - y'(T)) \leq 2\eta p(T) \sum_{t=T}^{\infty} A^{t-T} \hat{y} \leq 2\eta p(T)\hat{y}/(1 - \lambda) \leq 2\eta\alpha\beta/(1 - \lambda) \leq (\varepsilon/2)$, which contradicts (46). \square

Proof of Theorem 4: By Lemma 3, we have $p(t + 1) = p(t)A + w(t)b$ for $t \geq 0$. Denoting $p(t)\hat{y}$ by $k(t)$, and noting that $A\hat{y} < \lambda\hat{y}$, $b\hat{y} \leq 1$, we obtain

$$k(t + 1) \leq \lambda k(t) + w(t) \quad \text{for } t \geq 0. \quad (47)$$

Using Proposition 1, and noting that $v(t) = w(t)$ for $t \geq 0$ when $\delta = 1$, we can find a positive real number Q , such that $0 \leq w(t) \leq Q$ for $t \geq 0$. Using this in (47), we obtain $k(t + 1) \leq \lambda k(t) + Q$ for $t \geq 0$. Then, for $t \geq 1$, $k(t) \leq \lambda^t k(0) + Q[1 + \lambda + \dots + \lambda^{t-1}]$. Thus, $k(t) \leq k(0) + Q/(1 - \lambda)$ for $t \geq 0$. Since $\hat{y} \gg 0$, (7) must hold. The optimality of $\langle x(t), y(t), c(t) \rangle$ now follows from Theorem 3. \square

Acknowledgements

Earlier versions of the paper were presented at the Twelfth Econometric Society Meetings in Tucuman, Argentina and at the Theory Workshop of the University of Rochester. Acknowledgments are due to Lionel W. McKenzie and other participants for helpful comments. The current version has benefitted from suggestions by two referees of this journal. Research on this project was begun during the first author's sabbatical leave at Cornell University. The first author acknowledges partial support from an NSERC grant.

References

- Brock, W. A. (1970): "On the Existence of Weakly Maximal Programmes in a Multi-Sector Economy." *Review of Economic Studies* 37: 275–280.
- Brock, W. A., and Majumdar, M. (1988): "On Characterizing Optimal Competitive Programs in Terms of Decentralized Conditions." *Journal of Economic Theory* 45: 262–273.
- Dasgupta, S., and Mitra, T. (1988): "Characterization of Intertemporal Optimality in Terms of Decentralizable Conditions: the Discounted Case." *Journal of Economic Theory* 45: 274–287.
- (1990): "On Price Characterization of Optimal Plans in a Multi-Sector Economy." In *Economic Theory and Policy*, edited by B. Dutta, S. Gangopadhyay, D. Mookherjee, and D. Ray. Bombay: Oxford University Press.
- (1991): "Infinite Horizon Competitive Programs Are Optimal." Working Paper no. 91-05, Department of Economics, Dalhousie University, Halifax, N.S.
- (1994): "Transversality Conditions in Optimum Growth Models with or without Discounting: a Unified View." *Estudios de Economía* 21: 301–311.
- Dorfman, R., Samuelson, P., and Solow, R. (1958): *Linear Programming and Economic Analysis*. New York: McGraw-Hill.
- Gale, D. (1960): *The Theory of Linear Economic Models*. New York: McGraw-Hill.
- (1967): "On Optimal Development in a Multi-Sector Economy." *Review of Economic Studies* 34: 1–18.
- Gale, D., and Sutherland, W. R. S. (1968): "Analysis of a One Good Model of Economic Development." In *Mathematics of the Decision Sciences*, part 2, edited by G. B. Dantzig and A. F. Veinott, Jr. Providence, RI: American Mathematical Society.
- Koopmans, T. C. (1957): *Three Essays on the State of Economic Science*. New York: McGraw-Hill.
- Majumdar, M. (1974): "Efficient Programs in Infinite Dimensional Spaces: a Complete Characterization." *Journal of Economic Theory* 7: 355–369.
- (1988): "Decentralization in Infinite Horizon Economies: an Introduction." *Journal of Economic Theory* 45: 217–227.
- Malinvaud, E. (1953): "Capital Accumulation and Efficient Allocation of Resources." *Econometrica* 21: 233–268.
- McKenzie, L. W. (1986): "Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics." In *Handbook of Mathematical Economics*, vol. III, edited by K. J. Arrow and M. D. Intriligator. Amsterdam: Elsevier.
- Mitra, T., and Wan, H. Y., Jr. (1986): "On the Faustmann Solution to the Forest Management Problem." *Journal of Economic Theory* 40: 229–249.
- Peleg, B. (1970): Efficiency Prices for Optimal Consumption Plans, III. *Journal of Mathematical Analysis and Applications* 32: 630–638.
- (1974): "On Competitive Prices for Optimal Consumption Plans." *SIAM Journal of Applied Mathematics* 26: 239–253.
- Weitzman, M. L. (1973): "Duality Theory for Infinite Horizon Convex Models." *Management Science* 19: 783–789.

Addresses of authors: Swapan Dasgupta, Department of Economics, Dalhousie University, Halifax, Nova Scotia B3H 3J5, Canada; – Tapan Mitra, Department of Economics, Cornell University, Ithaca, NY 14853-7601, USA.